Science and Technology
RMUTT Journal

Vol.3 No.1 (2013) : 41 – 43
www.sci.rmutt.ac.th\stj
Online ISSN 2229-1547

A Note on an OpenProblem by B. Sroysang[1]

Julius Fergy T. Rabago

Department of Mathematics and Physics
College of Arts and Sciences, Central Luzon State University
Science City of Muñoz 3120, Nueva Ecija, Philippines
Email: julius_fergy.rabago@up.edu.ph

Abstract

In this short note, we answer an open problem posed by B. Sroysang [1]. That is, we show that the only solutions (x, y, z) in non-negative integers to the Diophantine equation $2^x + 31^y = z^2$ are $(3, 0, 3)$ and $(7, 2, 33)$.

Mathematics Subject Classification: 11D61.

Keywords: Exponential Diophantine equation, integral solutions.

1 Introduction

In [2], Sroysang showed that the Diophantine equation $8^x + 19^y = z^2$ has a unique solution $(x, y, z) = (1, 0, 3)$ in non-negative integers. At the end of his paper, he posed the following question "What is the set of all solutions (x, y, z) for the Diophantine equation $8^x + 17^y = z^2$ where x, y and z are non-negative integers?". The answer has been addressed by Rabago in [3]. He showed that the only solution to the Diophantine equation $8^x + 17^y = z^2$ are $(1, 0, 3), (1, 1, 5), (2, 1, 9)$, and $(3, 1, 23)$. On the other hand, Sroysang [4], studied the Diophantine equation $3^x + 5^y = z^2$ and showed that this equation has a unique non-negative integer solution $(1, 0, 2)$. In a paper by Suvarnamani, Singta and Chotchaisthit [5], the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have been shown to contain no non-negative integer solution. They used Catalan's conjecture, which is proven to be true by Mihailescu [6] in 2004, to prove their claim. In this note we answer the question raised by Sroysang in [1]. More precisely, we show that the Diophantine equation $2^x + 31^y = z^2$ has exactly two solutions in non-negative integers, *i.e.* $(x, y, z) = (3, 0, 3), (7, 2, 33)$.

[1]Received: February 25, 2013
Revised: September 13, 2013
Accepted: September 25, 2013

2 Main Results

We first prove a helpful Lemma.

Lemma 2.1. *Let n be a non-negative integer. Then,* $32|(31^{2n+1}+1)$ *and* $64|(31^{2n+1}+33)$, *or equivalently,* $31^{2n+1}+1=32l$ *for some odd natural number l.*

Proof. For $n=0$, we have $32|(31^1+1)$. Suppose $32|(31^{2k+1}+1)$ for some natural number k, *i.e.* $31^{2k+1}+1=32l$, l a natural number. Then $31^{2(k+1)+1}+1=961(31^{2k+1})+1=961(31^{2k+1}+1)-960=961(32l)-960=32(961l-30)$. Thus, $32|31^{2k+1}+1$.

On the other hand, for $n=0$, $64|(31^1+33)$. We assume that $64|(31^{2k+1}+33)$, where k is a natural number, *i.e.* $31^{2k+1}+33=64l$, for some natural number l. Hence, $31^{2(k+1)+1}+33=961(31^{2k+1})+33=961(31^{2k+1}+33)-31680=961(64l)-31680=64(961l-495)$. Therfore, $64|(31^{2k+1}+33)$. It follows that $64\nmid(31^{2k+1}+1)$. Here we conclude that $31^{2k+1}+1=32l$, for some odd natural number l, proving the theorem. □

Now we proceed to our main results.

Theorem 2.2. *The Diophantine equation $2^x+31^y=z^2$ has exactly two solutions in non-negative integers,* i.e. $(x,y,z)=(3,0,3),(7,2,33)$.

Proof. The case when $z=0$ is obvious so we only consider the following possibilities.

Case 1. $x=0$. Suppose $2^x+31^y=z^2$ is possible in non-negative integers x,y,z for $x=0$. Then we have $31^y=z^2-1=(z+1)(z-1)$. Letting $\alpha+\beta=y,\alpha<\beta$, we obtain $2=(z+1)-(z-1)=31^{\alpha-1}(31^{\beta-\alpha}-1)$. Hence, $\alpha=1$ and so, $31^{\beta-1}=3$, a contradiction.

Case 2. $y=0$. If $y=0$ then $z^2-1=(z+1)(z-1)=2^x$. So, $2=(z+1)-(z-1)=2^\alpha(2^{\beta-\alpha}-1)$, where $\alpha+\beta=x$, and $\alpha<\beta$. Hence, $\alpha=1$ and it follows that $\beta=2$. Thus, $x=3$ and $z=3$. This gives us a solution $(x,y,z)=(3,0,3)$ to $2^x+31^y=z^2$.

Case 3. $x,y,z>0$. We divide this case into two subcases.

Subcase 3.1 We first treat the case when $x=1$. So, suppose that $2^x+31^y=z^2$ is possible in non-negative integers x,y,z for $x=1$. Note that $31^y+2\equiv 3 \pmod 4$ if y is even, and $31^y+2\equiv 1 \pmod 4$ if y is odd. But, $z^2\equiv 0,1 \pmod 4$. So, y must be odd. Then we have, $31^{2n+1}+2=z^2$, where n is a natural number. So, it is either $z=4k+1$ or $z=4k+3$, $k=0$ or a natural number.

For $z=4k+1$, we have $31^{2n+1}+2=(4k+1)^2=16k^2+8k+1$. Then, $31^{2n+1}+1=8k(2k+1)$ and this implies that

$$k(2k+1)=\frac{31^{2n+1}+1}{8}=4\left(\frac{31^{2n+1}+1}{32}\right).$$

Note that by Lemma 2.1, $(31^{2n+1}+1)/32$ is odd. So, $k=4$ and $2k+1=(31^{2n+1}+1)/32$. Hence, $2(4)+1=(31^{2n+1}+1)/32$. It follows that, $31^{2n+1}=287$, a contradiction.

For $z = 4k+3$, we have $31^{2n+1}+2 = (4k+3)^2 = 16k^2+24k+9$. Then, $31^{2n+1}+1 = 16k^2+24k+8 = 8(2k^2+3k+1)$ and this implies that

$$(k+1)(2k+1) = \frac{31^{2n+1}+1}{8} = 4\left(\frac{31^{2n+1}+1}{32}\right).$$

Again, by Lemma 2.1, $(31^{2n+1}+1)/32$ is odd. Hence, $k = 3$ and $2k+1 = (31^{2n+1}+1)/32$ and this follows that $2(3)+1 = (31^{2n+1}+1)/32$. Thus, $31^{2n+1} = 223$, which is also a contradiction. Therefore, $31^y + 2 = z^2$, is impossible for non-negative integers y and z.

Subcase 3.2 For the case $x \geq 2$ we have $31^y + 2^x \equiv 1 \pmod 4$ if y is even and $31^y + 2^x \equiv 3 \pmod 4$ if y is odd. So, y must be even since $z^2 \equiv 0, 1 \pmod 4$. Let $y = 2n$, then $z^2 - (31^n)^2 = 2^x$. It follows that $2 \cdot 31^n = (z+31^n) - (z-31^n) = 2^\beta - 2^\alpha$, $\alpha + \beta = x$ and $\alpha < \beta$. Hence, $2^{\alpha-1}(2^{\beta-\alpha}-1) = 2^{\beta-1} - 2^{\alpha-1} = 31^n$. This implies that, $\alpha = 1$ and $2^{\beta-1} - 1 = 31^n$. But, the RHS can be expressed as $(32-1)^n = (2^5-1)^n = (2^{6-1}-1)^n$. Thus, $2^{\beta-1} - 1 = (2^{6-1}-1)^n$. Therefore, we can see immediately that $n = 1$ and $\beta = 6$. From these, we'll obtain $x = 7$ and $y = 2$. This gives us the value $z = 2^\alpha + 31^n = 2^1 + 31^1 = 33$. Here we conclude that $(x, y, z) = (7, 2, 33)$ is a solution of the Diophantine equation $2^x + 31^y = z^2$. Now, if $n > 1$ then $2^{\beta-1} - 31^n = 1$ is clearly impossible due to Catalan's conjecture. This completes the proof of the theorem. □

Corollary 2.3. *If n is a natural number different from one then, the Diophantine equation $2^x + 31^y = w^{2n}$ has no solution in non-negative integers.*

Proof. Let $n \neq 1$ be a natural number and suppose that the Diophantine equation $2^x + 31^y = (w^n)^2$ has a solution in non-negative integers. We let $z = w^n$, then we have $2^x + 31^y = z^2$. By Theorem 2.2, $z \in \{3, 33\}$. Hence, $w^n = 3$ or $w^n = 33$. These are possible only when $n = 1$, a contradiction. Thus, $2^x + 31^y = w^{2n}$ has no solution in non-negative integers. □

References

[1] B. Sroysang, On the Diophantine Equation $31^x + 32^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 609-612.

[2] B. Sroysang, More on the Diophantine Equation $8^x + 19^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 601-604.

[3] J. F. T. Rabago, On an Open Problem by B. Sroysang, *Konuralp Journal of Mathematics,* 2013, to appear.

[4] B. Sroysang, On the Diophantine Equation $3^x + 5^y = z^2$, *International Journal of Pure and Applied Mathematics,* **81**, 2012, no. 4, 605-608.

[5] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine Equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Technol. RMUTT J.,* **1**, 2011, 25-28.

[6] P. Mihailescu, Primary cycolotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.,* **27**, 2004, 167-195.