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Weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces

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Abstract

In this paper, we introduce an iterative method and prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of equilibrium problem with a relaxed monotone mapping and a common fixed point set nonspreading mappings. Using the result, we improve and unify several results in fixed point problems and equilibrium problems.

Keywords: Equilibrium problem, Fixed point problem, Nonexpansive mapping, Nonspreading mapping.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle,$$

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for all $x, y \in C$; see, for instance, [2, 5, 6, 15, 17]. On the other hand, a mapping $Q : C \to C$ is said to be quasi-nonexpansive if $F(Q) \neq \emptyset$ and

$$\|Qx - y\| \leq \|x - y\|,$$

for all $x \in C$ and $y \in F(Q)$, where $F(Q)$ is the set of fixed points of Q. If $T : C \to C$ is nonexpansive and the set $F(T)$ of fixed points of T is nonempty, then T is quasi-nonexpansive.

Recently, Kohsaka and Takahashi [10] introduced the following nonlinear mapping: Let E be a a Hilbert space and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \to C$ is said to be nonspreading if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

for all $x, y \in C$. We know in a Hilbert space that every firmly nonexpansive mapping is nonspreading and that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [10]. Let $A : C \to H$ be a mapping of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \;\; \forall u, v \in C.$$

A mapping $A : C \to H$ is called λ-inverse-strongly monotone if there exists a positive real number λ such that

$$\langle Au - Av, u - v\rangle \geq \lambda\|Ax - Ay\|^2 \;\; \forall u, v \in C.$$

A mapping $T : C \to H$ is said to be relaxed $\eta - \alpha$ monotone if there exist a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree p, that is, $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \quad \forall x, y \in C,$$

where $p > 1$ is a constant; see [4]. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$, T is said to be relaxed α-monotone. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 0$, T is said to be p-monotone; see [7, 16, 21]. In fact, in this case, if $p = 2$, then T is a k-strongly monotone mapping. Moreover, every monotone mapping is relaxed $\eta - \alpha$ monotone with $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha = 0$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \; \forall y \in C\}.$$

Many problems in physics, optimization, and economics require some elements of $EP(F)$, see [2, 3, 11, 18, 19, 20]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance [3, 18, 19, 20]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C,A)$. The generalized equilibrium problem for F and A is to find $x \in C$ such that

$$F(x,y) + \langle Ax, y-x\rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

Problem (1.1) was introduce by Takahashi and Takahashi [19] and the set of solution of (1.1) is denoted by $GEP(F,A)$. The generalized mixed equilibrium problem for F, ψ and A is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

Recently, Wang et al. [22] introduce the generalized mixed equilibrium problem with a relaxed monotone mapping. that is, to find $x \in C$ such that

$$F(x,y) + \big\langle Tx, \eta(y,x)\big\rangle + \langle Ax, y-x\rangle \geq 0 \quad \text{for all } y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $GEP(F,T)$.

On the other hand, Halpern [8] introduced the following iterative scheme for approximating a fixed point of T:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n \tag{1.4}$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0,1]$. Recently, Aoyama et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T_n x_n \tag{1.5}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself which satisfies the AKTT-condition, that is,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty. \tag{1.6}$$

They proved that the sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_n\}$.

In this paper, motivated by Plubtieng and Thammathiwat [14], Iemoto and Takahashi [9], Wang et al. [22], we introduce a new iterative sequence and prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of equilibrium problem with a relaxed monotone mapping and a common fixed point set nonspreading mappings.

2 Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

By definition of the metric projection P_C we have known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $y \in C$, $y = P_Cx$ if and only if $\langle x - y, y - z\rangle \geq 0$, $\forall z \in C$.

A space X is said to satisfy Opial's condition [12] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x$$

and

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

Lemma 2.1. [10] *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

In order to prove the main result, we shall use the following lemmas in the sequel.

Lemma 2.2. [9] *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \to C$ a nonspreading mapping with $F(S) \neq \emptyset$. Then S is semiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Sx_n \to 0$ imply $u \in F(S)$.*

Lemma 2.3. [13] *Let C be a nonempty bounded closed convex subset of Hilbert space E and $\{T_n\}$ a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l\to\infty} \rho_l^k = 0 \tag{2.1}$$

where $\rho_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be a mapping from C in to itself defined by

$$Tx = \lim_{n\to\infty} T_n x, \quad for\ all\ x \in C.$$

Then $\limsup_{n\to\infty}\{\|Tz - T_n z\| : z \in C\} = 0$.

In fact, Aoyama et al. [1] proved Lemma 2.3 in case the sequence $\{T_n\}$ satisfies the AKTT-condition. We note that if a sequence $\{T_n\}$ satisfies the AKTT-condition then $\{T_n\}$ satisfies the condition (2.1).

3 Weak convergence theorem

In this section, we prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $\{T_n\}$ be the sequences of nonexpansive mappings of C into itself such that $F := F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S T_n x_n, \quad n \geq 0. \tag{3.1}$$

Suppose that $\{T_n\}$ satisfy the AKTT-condition, T be the mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$ such that $F(T) = \cap_{n=1}^{\infty} F(T_n)$ and suppose that for any $v \in F$,

$$\|T_n x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_n x_n\|^2 + c_n,$$

where $\lim_{n\to\infty} c_n = 0$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F$.

Proof. Take a point $v \in F$ and put $y_n = T_n x_n$. We shall show that the sequences $\{x_n\}$ is bounded. First, we note that

$$\begin{aligned} \|Sy_n - v\| &\leq \|y_n - v\| \\ &= \|T_n x_n - v\| \\ &\leq \|x_n - v\| \end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1}-v\|^2 &= \|\alpha_n x_n+(1-\alpha_n)Sy_n-v\|^2\\
&= \|\alpha_n(x_n-v)+(1-\alpha_n)(Sy_n-v)\|^2\\
&= \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|Sy_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\
&\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|y_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\
&\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|x_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\
&= \|x_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\
&\le \|x_n-v\|^2. \qquad (3.2)
\end{aligned}$$

Then $\{\|x_{n+1}-v\|\}$ is a decreasing sequence and therefore $\lim_{n\to\infty}\|x_n-v\|$ exists. This implies that $\{x_n\},\{T_nx_n\},\{y_n\}$ and $\{Sy_n\}$ are bounded. By our assumption, we have

$$\|T_nx_n-v\|^2\le\|x_n-v\|^2-\|x_n-T_nx_n\|^2+c_n,$$

where $\lim_{n\to\infty}c_n=0$. Thus, we note that

$$\begin{aligned}
\|x_{n+1}-v\|^2 &= \|\alpha_n x_n+(1-\alpha_n)Sy_n-v\|^2\\
&= \|\alpha_n(x_n-v)+(1-\alpha_n)(Sy_n-v)\|^2\\
&\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|Sy_n-v\|^2\\
&\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|y_n-v\|^2\\
&= \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|T_nx_n-v\|^2\\
&\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\big(\|x_n-v\|^2-\|x_n-T_nx_n\|^2+c_n\big) \qquad (3.3)
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)\|x_n-T_nx_n\|^2 &\le \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|x_n-v\|^2\\
&\quad +(1-\alpha_n)c_n-\|x_{n+1}-v\|^2\\
&= \|x_n-v\|^2-\|x_{n+1}-v\|^2+(1-\alpha_n)c_n.
\end{aligned}$$

Since $0<a\le\alpha_n\le b<1, c_n\to 0$ and $\lim_{n\to\infty}\|x_n-v\|^2=\lim_{n\to\infty}\|x_{n+1}-v\|^2$, it follows that

$$\|x_n-T_nx_n\|=\|x_n-y_n\|\to 0. \qquad (3.4)$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{z}$. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup\hat{z}$. By Lemma 2.2, we have $\hat{z}\in F(S)$. Since $\lim_{n\to\infty}\|x_n-y_n\|\to 0$ and $y_{n_i}\rightharpoonup\hat{z}$, we get $x_{n_i}\rightharpoonup\hat{z}$.

We shall show that $\hat{z} \in F(T)$. From $\|T_n x_n - x_n\| \to 0$ and the AKTT-condition, we have $\|Tx_n - x_n\| \leq \|Tx_n - T_n x_n\| + \|T_n x_n - x_n\| \to 0$. We next show that $\hat{z} \in F(T)$. Assume $\hat{z} \notin F(T)$. Since $x_{n_i} \rightharpoonup \hat{z}$ and $\hat{z} \neq T\hat{z}$. By the Opials condition, we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_{n_i} - \hat{z}\| &< \liminf_{n\to\infty} \|x_{n_i} - T\hat{z}\| \\ &\leq \liminf_{n\to\infty} \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - T\hat{z}\| \\ &\leq \liminf_{n\to\infty} \|x_{n_i} - \hat{z}\|.\end{aligned}$$

This is a contradiction. So, we get $\hat{z} \in F(T)$. Hence $\hat{z} \in F$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\bar{z}$. We may show that $\hat{z} = \tilde{z}$, suppose not. Since $\lim_{n\to\infty} \|x_n - v\|$ exists for all $v \in F$, it follows by the Opial's condition that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \hat{z}\| &= \liminf_{i\to\infty} \|x_{n_i} - \hat{z}\| < \liminf_{i\to\infty} \|x_{n_i} - \bar{z}\| = \lim_{n\to\infty} \|x_n - \bar{z}\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - \bar{z}\| < \liminf_{k\to\infty} \|x_{n_k} - \hat{z}\| = \lim_{n\to\infty} \|x_n - \hat{z}\|.\end{aligned}$$

This is a contradiction. Thus, we have $\hat{z} = \tilde{z}$. This implies that $\{x_n\}$ converges weakly to $\hat{z} \in F$. This completes the proof. □

Corollary 3.2. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $\{T_n\}$ be the sequences of firmly nonexpansive mappings of C into itself such that $F := F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ is nonempty. Let $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Let $\{x_n\}$ be a sequence defined by $x_0 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S T_n x_n, \quad n \geq 0. \tag{3.5}$$

Suppose that $\{T_n\}$ satisfy the AKTT-condition and T be the mappings of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$ and suppose that $F(T) = \cap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F$.

Proof. Since $\{T_n\}$ is firmly nonexpansive, it follows that

$$\begin{aligned}\|T_n x_n - v\|^2 &= \|T_n x_n - T_n v\|^2 \\ &\leq \langle T_n x_n - v, x_n - v\rangle \\ &= \frac{1}{2}\big(\|T_n x_n - v\|^2 + \|x_n - v\|^2 - \|x_n - T_n x_n\|^2\big),\end{aligned}$$

for all $v \in F$ and hence $\|T_n x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_n x_n\|^2$. This implies that $\{T_n x_n\}$ satisfying condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. □

4 Applications

In this section, using Theorem 3.1, we prove weak convergence theorem for finding a common element of the set of solutions of equilibrium problem with a relaxed monotone mapping and the fixed point set of a nonspreading mapping in Hilbert space. Before, proving our theorems, we need the following lemmas. For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$.

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$.

(A3) for each $x, y, z \in C$. $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$.

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 4.1. [2] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [3].

Lemma 4.2. [3] *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) T_r is single- valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 4.3. [22] *Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to H$ be an η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping and let Φ be a bifunction from $C \times C \to R$ satisfying (A1), (A2), and (A4). Let $r > 0$ and define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \Phi(z, y) + \langle Tz, \eta(y, z)\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Assume that

(i) $\eta(x, y) + \eta(y, x) = 0$, *for all* $x, y \in C$;

(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex and lower semicontinuous and the mapping $x \mapsto \langle Tu, \eta(v, x)\rangle$ is lower semicontinuous;

(iii) $\alpha : H \to \mathbb{R}$ *is weakly lower semicontinuous;*

(iv) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.

Then, the following holds:

(1) T_r *is single- valued;*

(2) T_r *is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(3) $F(T_r) = EP(\Phi, T)$;

(4) $EP(\Phi, T)$ *is closed and convex.*

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\psi : C \times C \to \mathbb{R}$ satisfying (A1)-(A4). Let T be η-hemicontinuous and relaxed $\eta - \alpha$ monotone mapping of C into H and let S be a nonspreading mapping of C into itself such that $F := F(S) \cap EP(\psi, T) \neq \emptyset$. Suppose $x_0 = x \in C$ and define the sequence $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} \psi(y_n, y) + \big\langle Ty_n, \eta(y, y_n)\big\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{r_n\} \in (0, \infty)$ with $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F$.

Proof. Setting $T_n \equiv T_{r_n}$ in Theorem 3.1 and putting $y_n = T_{r_n} x_n$. Let $v \in F$ and let B be a bounded subset of C. For $n \in \mathbb{N}$, let $z_n = T_{r_n} z$. we first prove that

$$\sum_{n=1}^{\infty} \sup \Big\{ \|T_{r_{n+1}} z - T_{r_n} z\| : z \in B \Big\} < \infty \tag{4.1}$$

for any bounded subset B of C. We note that

$$\psi(z_n, y) + \big\langle T z_n, \eta(y, z_n) \big\rangle + \frac{1}{r_n} \langle y - z_n, z_n - z \rangle \geq 0 \tag{4.2}$$

for all $y \in C$ and

$$\psi(z_{n+1}, y) + \big\langle T z_{n+1}, \eta(y, z_{n+1}) \big\rangle + \frac{1}{r_{n+1}} \langle y - z_{n+1}, z_{n+1} - z \rangle \geq 0 \tag{4.3}$$

for all $y \in C$. Setting $y = z_{n+1}$ in (4.2) and $y = z_n$ in (4.3), we have

$$\psi(z_n, z_{n+1}) + \big\langle T z_n, \eta(z_{n+1}, z_n) \big\rangle + \frac{1}{r_n} \langle z_{n+1} - z_n, z_n - z \rangle \geq 0$$

and

$$\psi(z_{n+1}, z_n) + \big\langle T z_{n+1}, \eta(z_n, z_{n+1}) \big\rangle + \frac{1}{r_{n+1}} \langle z_n - z_{n+1}, z_{n+1} - z \rangle \geq 0.$$

Adding the two inequalities and by (A2), we have

$$\big\langle T z_n - T z_{n+1}, \eta(z_{n+1}, z_n) \big\rangle + \left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z_n}{r_{n+1}} \right\rangle \geq 0.$$

Thus, we have

$$\left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z_n}{r_{n+1}} \right\rangle \geq \big\langle T z_{n+1} - T z_n, \eta(z_{n+1}, z_n) \big\rangle$$

and hence

$$\begin{aligned} \langle z_{n+1} - z_n, z_n - z_{n+1} \rangle \quad &+ \quad \left\langle z_{n+1} - z_n, (1 - \frac{r_n}{r_{n+1}})(z_{n+1} - z) \right\rangle \\ &\geq \quad \big\langle T z_{n+1} - T z_n, \eta(z_{n+1}, z_n) \big\rangle \end{aligned} \tag{4.4}$$

Since T is relaxed $\eta - \alpha$ monotone mapping and $r > 0$, we have

$$-\|z_{n+1} - z_n\|^2 + \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right)(z_{n+1} - z) \right\rangle \quad \geq \quad r\alpha(z_{n+1} - z_n). \tag{4.5}$$

Similarly, by exchanging the position of z_{n+1} and z_n in (4.4), we get

$$\begin{aligned}\langle z_n - z_{n+1}, z_{n+1} - z_n\rangle &+ \Big\langle z_n - z_{n+1}, (1 - \frac{r_{n+1}}{r_n})(z_n - z)\Big\rangle \\ &\geq \big\langle Tz_n - Tz_{n+1}, \eta(z_n, z_{n+1})\big\rangle. \end{aligned} \tag{4.6}$$

and hence

$$-\|z_n - z_{n+1}\|^2 + \Big\langle z_n - z_{n+1}, \Big(1 - \frac{r_{n+1}}{r_n}\Big)(z_n - z)\Big\rangle \ \geq\ r\alpha(z_n - z_{n+1}). \tag{4.7}$$

Adding (4.5) and (4.7), we have

$$\begin{aligned} 2\|z_{n+1} - z_n\|^2 &\leq \Big\langle z_{n+1} - z_n, \Big(1 - \frac{r_n}{r_{n+1}}\Big)(z_{n+1} - z)\Big\rangle \\ &\quad + \Big\langle z_{n+1} - z_n, \Big(\frac{r_{n+1}}{r_n} - 1\Big)(z_n - z)\Big\rangle \\ &\leq \|z_{n+1} - z_n\|\Big|1 - \frac{r_n}{r_{n+1}}\Big|\|z_{n+1} - z\| \\ &\quad + \|z_{n+1} - z_n\|\Big|1 - \frac{r_{n+1}}{r_n}\Big|\|z_n - z\|. \end{aligned}$$

Thus $\|z_{n+1} - z_n\| \leq \frac{1}{2}\Big[\Big|1 - \frac{r_n}{r_{n+1}}\Big|\|z_{n+1} - z\| + \Big|1 - \frac{r_{n+1}}{r_n}\Big|\|z_n - z\|\Big]$. Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \|T_{r_{n+1}}z - T_{r_n}z\| &= \|z_{n+1} - z_n\| \\ &\leq \frac{1}{2}\Big[\frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T_{r_{n+1}}z - z\| + \frac{1}{r_n}|r_n - r_{n+1}|\|T_{r_n}z - z\|\Big] \\ &\leq \frac{1}{2}\Big[\frac{1}{b}|r_{n+1} - r_n|\|T_{r_{n+1}}z - z\| + \frac{1}{b}|r_n - r_{n+1}|\|T_{r_n}z - z\|\Big]. \end{aligned}$$

Let $u \in EP(\psi, T)$ and $M = \sup\big\{\|z - u\| : z \in B\big\}$. Then

$$\begin{aligned} \|T_{r_{n+1}}z - z\| &\leq \|T_{r_{n+1}}z - u\| + \|u - z\| \\ &= \|T_{r_{n+1}}z - T_{r_{n+1}}u\| + \|u - z\| \\ &\leq 2\|z - u\|. \end{aligned}$$

Similarly, we note that $\|T_{r_n}z - z\| \leq 2\|z-u\|$. Thus, we have

$$\begin{aligned}\|T_{r_{n+1}}z - T_{r_n}z\| &\leq \frac{1}{2}\left[\frac{1}{b}|r_{n+1}-r_n|\|T_{r_{n+1}}z - z\| + \frac{1}{b}|r_n - r_{n+1}|\|T_{r_n}z - z\|\right] \\ &\leq \frac{1}{2}\left[\frac{4M}{b}|r_{n+1}-r_n|\right] \\ &= \frac{2M}{b}|r_{n+1}-r_n|.\end{aligned}$$

Hence $\sup\left\{\|T_{r_{n+1}}z - T_{r_n}z\| : z \in B\right\} \leq \frac{2M}{b}|r_{n+1}-r_n|$. Since $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$, we obtain $\sum_{n=1}^{\infty}\sup\left\{\|T_{r_{n+1}}z - T_{r_n}z\| : z \in B\right\} < \infty$. By Lemma 2.3, we define a mapping T by $Tx = \lim_{n\to\infty} T_{r_n}x$ for all $x \in C$.

Next, we prove that $F(T) = \cap_{n=1}^{\infty}F(T_{r_n})$. It easy to see that $\cap_{n=1}^{\infty}F(T_{r_n}) \subset F(T)$. Let $w \in F(T)$. For $n \in \mathbb{N}$, let $w_n = T_{r_n}w$. Then

$$\psi(w_n, y) + \big\langle Tw_n, \eta(y, w_n)\big\rangle + \frac{1}{r_n}\langle y - w_n, w_n - w\rangle \geq 0$$

for all $y \in C$. By (A2), we obtain $\frac{1}{r_n}\langle y - w_n, w_n - w\rangle \geq \psi(y, w_n) + \big\langle Tw_n, \eta(w_n, y)\big\rangle$ for all $y \in C$. Since $w_n \to w$ and from (A4), we have $0 \geq \psi(y, w) + \big\langle Tw, \eta(w, y)\big\rangle$ for all $y \in C$. Put $u_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. Thus, we note that

$$\begin{aligned} 0 &= \psi(u_t, u_t) + \big\langle Tw, \eta(u_t, u_t\big\rangle \\ &= \psi(ty + (1-t)w, ty + (1-t)w) + \big\langle Tw, \eta(ty + (1-t)w, u_t)\big\rangle \\ &\leq t\psi(ty + (1-t)w, y) + (1-t)\psi(ty + (1-t)w, w) + t\big\langle Tw, \eta(y, u_t)\big\rangle \\ &\quad +(1-t)\big\langle Tw, \eta(w, u_t)\big\rangle \\ &= t\big[\psi(ty + (1-t)w, y) + \big\langle Tw, \eta(y, u_t)\big\rangle\big] \\ &\quad +(1-t)\big[\psi(ty + (1-t)w, w) + \big\langle Tw, \eta(w, u_t)\big\rangle\big] \\ &\leq t\big[\psi(ty + (1-t)w, y) + \big\langle Tw, \eta(y, u_t)\big\rangle\big]. \end{aligned}$$

So, $\psi(ty + (1-t)w, y) + \big\langle Tw, \eta(y, u_t)\big\rangle \geq 0$ for all $y \in C$. Letting $t \to 0^+$ and using (A3), we obtain $\psi(w, y) + \big\langle Tw, \eta(y, w)\big\rangle \geq 0$ for all $y \in C$. Thus $w \in EP(\psi, T)$. It follows that $w \in \cap_{n=1}^{\infty}F(T_{r_n})$ and it easy to see that

$$\|T_{r_n}x_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - T_{r_n}x_n\|^2.$$

Thus $\{T_{r_n}x_n\}$ satisfying condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. □

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References

[1] K.Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. 67 (2007) 2350-2360.

[2] E. Blum and W. Oettli, From optimization and variationnal inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[3] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117-136.

[4] Y. P. Fang and N. J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, J. Optim. Theo. Appl. 118, no. 2 (2003) 327-338.

[5] K. Goebel and W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[6] K. Geobel and S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker inc. New York, 1984.

[7] D. Goeleven and D. Motreanu, Eigenvalue and dynamic problems for variational and hemivariational inequalities, Comm. Appl. Nonl. Anal. 3, no. 4(1996) 121.

[8] B. Halpern, Fixed point of nonexpanding maps, Bull. amer. Math. Soc. 73 (1967) 957-961.

[9] S. Iemoto and W. Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.03.064.

[10] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. 91(2008) 166-177.

[11] A. Moudafi and M. Thera, Proximal and Dynamical Approaches to Equilibrium Problems, in: Lecture Notes in Economics and Mathematical Systems, 477 (1999) 187-201.

[12] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73(1967) 595-597.

[13] S. Plubtieng and T. Thammathiwat, A viscosity approximation method for finding a common fixed point of nonexpansive and firmly nonexpansive mappings in hilbert spaces. Thai. J. Math. 6 (2008) 377-390.

[14] S. Plubtieng and T. Thammathiwat, A viscosity approximation method for finding a common solution of fixed point and equilibrium problems in Hilbert spaces. J. Glob. Optium, (2010) doi:10.1007/s10898-010-9583-z.

[15] S. Reich and D. Shoikhet, Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces, Imperial College Press, London, 2005.

[16] A. H. Siddiqi, Q. H. Ansari, and K. R. Kazmi, On nonlinear variational inequalities, Indian Journal of Pure and Applied Mathematics. 25, no. 9 (1994) 969-973.

[17] W. Takahashi, Nonlinear Functional Analysis-Fixed Point Theory and its Applications, Yokohama Publishers, Yokohama, 2000.

[18] A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, J. Optim. Theory Appl. 133 (2007) 359-370.

[19] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506-515.

[20] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025-1033.

[21] R. U. Verma, Nonlinear variational inequalities on convex subsets of Banach spaces, Appl. Math. Letters. 10, no. 4 (1997) 25-27.

[22] S. Wang, G. Marino, F. Wang, Strong Convergence Theorems for a Generalized Equilibrium Problem with a Relaxed Monotone Mapping and a Countable Family of Nonexpansive Mappings in a Hilbert Space, Fixed Point Theory and Applications (2010), doi:10.1155/2010/230304.